Low frequency dispersive estimates for the Schrödinger group in higher dimensions

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Abstract

For a large class of real-valued potentials, $V(x)$, $x \in \mathbf{R}^n$, $n \geq 4$, we prove dispersive estimates for the low frequency part of $e^{it(-\Delta+V)}P_{ac}$, provided the zero is neither an eigenvalue nor a resonance of $-\Delta+V$, where P_{ac} is the spectral projection onto the absolutely continuous spectrum of $-\Delta+V$. This class includes potentials $V \in L^\infty(\mathbf{R}^n)$ satisfying $V(x) = O\left(\langle x\rangle^{-(n+2)/2-\epsilon}\right)$, $\epsilon > 0$. As a consequence, we extend the results in [4] to a larger class of potentials.

1 Introduction and statement of results

Let $V \in L^\infty(\mathbf{R}^n)$, $n \geq 4$, be a real-valued function satisfying

$$|V(x)| \leq C\langle x\rangle^{-\delta}, \quad \forall x \in \mathbf{R}^n, \tag{1.1}$$

with constants $C > 0$, $\delta > (n+2)/2$. Denote by G_0 and G the self-adjoint realizations of the operators $-\Delta$ and $-\Delta+V$ on $L^2(\mathbf{R}^n)$, respectively. It is well known that the absolutely continuous spectrums of the operators G_0 and G coincide with the interval $[0,+\infty)$, and that G has no embedded strictly positive eigenvalues nor strictly positive resonances. However, G may have in general a finite number of non-positive eigenvalues and that the zero may be a resonance. We will say that the zero is a regular point for G if it is neither an eigenvalue nor a resonance in the sense that the operator $1 - V\Delta^{-1}$ is invertible on L^1 with a bounded inverse. Let P_{ac} denote the spectral projection onto the absolutely continuous spectrum of G. When $n \geq 3$, Journé, Sofer and Sogge [4] proved the following dispersive estimate

$$\left\|e^{itG}P_{ac}\right\|_{L^1\to L^\infty} \leq C|t|^{-n/2}, \quad t \neq 0, \tag{1.2}$$

provided the zero is neither an eigenvalue nor a resonance, for potentials satisfying (1.1) with $\delta > n+4$ as well as the condition

$$\widehat{V} \in L^1. \tag{1.3}$$

This was later improved by Yajima [9] for potentials satisfying (1.1) with $\delta > n+2$. When $n = 3$, the estimate (1.2) in fact holds without (1.3). In this case, it was proved in [2] for potentials satisfying (1.1) with $\delta > 3$ and was later improved in [6] and [10] for potentials satisfying (1.1) with $\delta > 5/2$. Goldberg [1] has recently showed that (1.2) holds for potentials $V \in L^{3/2-\epsilon} \cap L^{3/2+\epsilon}$, $0 < \epsilon \ll 1$, which includes potentials satisfying (1.1) with $\delta > 2$. When $n = 2$, (1.2) is proved by Schlag [5] for potentials satisfying (1.1) with $\delta > 3$.

Given any $a > 0$, set $\chi_a(\sigma) = \chi_1(\sigma/a)$, where $\chi_1 \in C^\infty(\mathbf{R})$, $\chi_1(\sigma) = 0$ for $\sigma \leq 1$, $\chi_1(\sigma) = 1$ for $\sigma \geq 2$. Set $\eta_a = \chi(1-\chi_a)$, where χ denotes the characteristic function of the interval $[0,+\infty)$. Clearly, $\eta_a(G) + \chi_a(G) = P_{ac}$. When $n \geq 4$, dispersive estimates with loss of $(n-3)/2$ derivatives for the operator $e^{itG}\chi_a(G)$, $\forall a > 0$, have been recently proved in

[7] under the assumption (1.1), only. The loss of derivatives in this case is a high frequency phenomenon and cannot be avoided unless one imposes some regularity condition on the potential (see [3]). The condition (1.3) in [4] plays this role but it seems too strong. The natural conjecture would be that we have dispersive estimates for $e^{itG}\chi_a(G)$ with loss of ν derivatives, $0 \le \nu \le (n-3)/2$, provided $V \in C^{(n-3)/2-\nu}(\mathbf{R}^n)$ (with a suitable decay at infinity). It turns out that no regularity on the potential is needed in order to get dispersive estimates for the low frequency part $e^{itG}\eta_a(G)$, $a > 0$ small. One just needs some decay at infinity. In fact, the low frequency analysis turns out to be easier in dimensions $n \ge 4$ compared with the cases of $n = 2$ and $n = 3$, and can be carried out for a larger class of potentials satisfying (with some $0 < \epsilon \ll 1$)

$$\sup_{y \in \mathbf{R}^n} \int_{\mathbf{R}^n} \left(|x-y|^{-n+2} + |x-y|^{-(n-2)/2+\epsilon} \right) |V(x)| dx \le C < +\infty. \tag{1.4}$$

Clearly, (1.4) is fulfilled for potentials satisfying (1.1). Our main result is the following

Theorem 1.1 *Let $n \ge 4$, let V satisfy (1.4) and assume that the zero is a regular point for G. Then, there exists a constant $a_0 > 0$ so that for $0 < a \le a_0$ we have the estimate*

$$\left\| e^{itG}\eta_a(G) \right\|_{L^1 \to L^\infty} \le C|t|^{-n/2}, \quad t \neq 0. \tag{1.5}$$

Remark 1. We expect that (1.5) holds true for the larger class of potentials satisfying

$$\sup_{y \in \mathbf{R}^n} \int_{\mathbf{R}^n} \left(|x-y|^{-n+2} + |x-y|^{-(n-1)/2} \right) |V(x)| dx \le C < +\infty, \tag{1.6}$$

but the proof in this case would require a different approach.

Combining (1.5) with the estimates of [7], we obtain the following

Theorem 1.2 *Let $n \ge 4$, let V satisfy (1.1) and assume that the zero is a regular point for G. Then, we have the estimates, $\forall t \neq 0$, $0 < \epsilon \ll 1$,*

$$\left\| e^{itG} P_{ac} f \right\|_{L^\infty} \le C|t|^{-n/2} \left\| \langle G \rangle^{(n-3)/4} f \right\|_{L^1}, \tag{1.7}$$

$$\left\| e^{itG} P_{ac} f \right\|_{L^\infty} \le C_\epsilon |t|^{-n/2} \left\| \langle x \rangle^{n/2+\epsilon} f \right\|_{L^2}. \tag{1.8}$$

Remark 2. The proof in [7] is based on uniform estimates for the operator $\psi(h^2 G)$, $0 < h \le 1$, $\psi \in C_0^\infty((0,+\infty))$ (see Lemma 2.2 of [7] or Lemma 2.3 of [8]). In the proof of this lemma (which is given in [8]), however, there is a mistake. That is why, we will give a new proof in Appendix 1 of the present paper.

Remark 3. We conjecture that the estimates (1.7) and (1.8) hold true for potentials satisfying (1.1) with $\delta > (n+1)/2$.

Theorem 1.1 also allows to extend the results in [4] to a larger class of potentials. More precisely, we have the following

Theorem 1.3 *Let $n \ge 4$, let V satisfy (1.1) with $\delta > n-1$ as well as (1.3), and assume that the zero is a regular point for G. Then, the estimate (1.2) holds true.*

Theorem 1.3 follows from (1.5) and the dispersive estimate for $e^{itG}\chi_a(G)$ proved in Appendix 2.

To prove (1.5) we adapt the *semi-classical* approach of [7] based on the *semi-classical* version of Duhamel's formula (which in our case is of the form (3.4) or (3.5)). While in [7] the estimates had to be uniform with respect to the semi-classical parameter $0 < h \le 1$, in the case of low frequency we need to make them uniform for $h \gg 1$ (see (3.1)). This,

however, turns out to be easier (when $n \geq 4$) as we can absorb the remaining terms taking h big enough (see Section 3). That is why, we do not need any more to work on weighted L^2 spaces (as in [7]), which in turn allows to cover a much larger class of potentials. As mentioned in Remark 1, the natural class of potentials for which the low frequency analysis works out (for $n \geq 4$) is given by (1.6), and the fact that the crucial Proposition 2.1 below holds true under (1.6) is a strong indication for that. In fact, (1.4) is used in the proof of Proposition 2.3, only.

2 Preliminary estimates

Let $\psi \in C_0^\infty((0, +\infty))$. We will first prove the following

Proposition 2.1 *Let $n \geq 4$, let V satisfy (1.6) and assume that the zero is a regular point for G. Then, there exist positive constants C, β and h_0 so that the following estimates hold*

$$\left\| \psi(h^2 G_0) \right\|_{L^1 \to L^1} \leq C, \quad h > 0, \tag{2.1}$$

$$\left\| \psi(h^2 G) \right\|_{L^1 \to L^1} \leq C, \quad h \geq h_0, \tag{2.2}$$

$$\left\| \psi(h^2 G) - \psi(h^2 G_0) T \right\|_{L^1 \to L^1} \leq C h^{-\beta}, \quad h \geq h_0, \tag{2.3}$$

where the operator

$$T = \left(1 - V\Delta^{-1}\right)^{-1} : L^1 \to L^1 \tag{2.4}$$

is bounded by assumption.

Proof. Set $\varphi(\lambda) = \psi(\lambda^2)$. We are going to take advantage of the formula

$$\psi(h^2 G) = \frac{2}{\pi} \int_{\mathbf{C}} \frac{\partial \widetilde{\varphi}}{\partial \bar{z}}(z)(h^2 G - z^2)^{-1} z L(dz), \tag{2.5}$$

where $L(dz)$ denotes the Lebesgue measure on $\mathbf{C}$, $\widetilde{\varphi} \in C_0^\infty(\mathbf{C})$ is an almost analytic continuation of φ supported in a small complex neighbourhood of $\operatorname{supp} \varphi$ and satisfying

$$\left| \frac{\partial \widetilde{\varphi}}{\partial \bar{z}}(z) \right| \leq C_N |\operatorname{Im} z|^N, \quad \forall N \geq 1.$$

For $\pm \operatorname{Im} z > 0$, denote

$$\mathcal{R}_{0,h}^{\pm}(z) = (h^2 G_0 - z^2)^{-1}, \quad \mathcal{R}_h^{\pm}(z) = (h^2 G - z^2)^{-1}.$$

The kernel of the operator $\mathcal{R}_{0,h}^{\pm}(z)$ is of the form $R_h^{\pm}(|x-y|, z)$, where

$$R_h^{\pm}(\sigma, z) = \pm h^{-2} \frac{i\sigma^{-2\nu}}{4(2\pi)^\nu} \mathcal{H}_\nu^{\pm}(\sigma z/h) = h^{-n} R_1^{\pm}(\sigma h^{-1}, z),$$

where $\nu = (n-2)/2$, $\mathcal{H}_\nu^{\pm}(\lambda) = \lambda^\nu H_\nu^{\pm}(\lambda)$, $H_\nu^{\pm}$ being the outgoing and incoming Henkel functions of order ν. It is well known that these functions satisfy the bound

$$\left| \mathcal{H}_\nu^{\pm}(\lambda) \right| \leq C \langle \lambda \rangle^{(n-3)/2} e^{-|\operatorname{Im} \lambda|}, \quad \forall \lambda, \pm \operatorname{Im} \lambda \geq 0, \tag{2.6}$$

while near $\lambda = 0$ they are of the form

$$\mathcal{H}_\nu^{\pm}(\lambda) = a_{\nu,1}^{\pm}(\lambda) + \lambda^{n-2} \log \lambda \, a_{\nu,2}^{\pm}(\lambda), \tag{2.7}$$

where $a^{\pm}_{\nu,j}$ are analytic functions, $a^{\pm}_{\nu,2} \equiv 0$ if n is odd. By (2.6) and (2.7), we have

$$\left|\mathcal{H}^{\pm}_{\nu}(\lambda) - \mathcal{H}^{\pm}_{\nu}(0)\right| \le C|\lambda|^{1/2}\langle\lambda\rangle^{(n-4)/2}, \quad \forall\lambda,\ \pm\mathrm{Im}\,\lambda \ge 0. \tag{2.8}$$

Hence, the functions $R^{\pm}_{h}$ satisfy the bounds (for $z \in \mathbf{C}^{\pm}_{\varphi} := \{z \in \mathrm{supp}\,\widetilde{\varphi},\ \pm\mathrm{Im}\,z \ge 0\}$, $\sigma > 0$, $h \ge 1$)

$$\left|R^{\pm}_{h}(\sigma, z)\right| \le Ch^{-2}\left(\sigma^{-n+2} + \sigma^{-(n-1)/2}\right), \tag{2.9}$$

$$\left|R^{\pm}_{h}(\sigma, z) - R^{\pm}_{h}(\sigma, 0)\right| \le Ch^{-5/2}\left(\sigma^{-n+5/2} + \sigma^{-(n-1)/2}\right). \tag{2.10}$$

Using the above bounds we will prove the following

Lemma 2.2 *For $z \in \mathbf{C}^{\pm}_{\varphi}$, we have*

$$\|V\mathcal{R}_{0,h}(z)\|_{L^1\to L^1} \le Ch^{-2}, \quad h \ge 1, \tag{2.11}$$

$$\|V\mathcal{R}_{0,h}(z) - V\mathcal{R}_{0,h}(0)\|_{L^1\to L^1} \le Ch^{-5/2}, \quad h \ge 1, \tag{2.12}$$

$$\|V\mathcal{R}_{h}(z)\|_{L^1\to L^1} \le Ch^{-2}, \quad h \ge h_0, \tag{2.13}$$

$$\left\|\mathcal{R}^{\pm}_{0,h}(z)\right\|_{L^1\to L^1} \le C|\mathrm{Im}\,z|^{-q}, \quad h > 0,\ \mathrm{Im}\,z \ne 0, \tag{2.14}$$

$$\left\|\mathcal{R}^{\pm}_{h}(z)\right\|_{L^1\to L^1} \le C|\mathrm{Im}\,z|^{-q}, \quad h \ge h_0,\ \mathrm{Im}\,z \ne 0, \tag{2.15}$$

with constants $C, q, h_0 > 0$ independent of z and h.

Proof. In view of (2.9), the norm in the LHS of (2.11) is upper bounded by

$$\sup_{y\in\mathbf{R}^n}\int_{\mathbf{R}^n} |V(x)|\left|R^{\pm}_{h}(|x-y|, z)\right| dx$$

$$\le Ch^{-2}\sup_{y\in\mathbf{R}^n}\int_{\mathbf{R}^n} |V(x)|\left(|x-y|^{-n+2} + |x-y|^{-(n-1)/2}\right) dx \le Ch^{-2}.$$

The estimate (2.12) follows in the same way using (2.10). To prove (2.14), we use (2.6) to get (for $z \in \mathbf{C}^{\pm}_{\varphi}$, $\mathrm{Im}\,z \ne 0$)

$$\left|R^{\pm}_{1}(\sigma, z)\right| \le C\sigma^{-2\nu}\langle\sigma\rangle^{(n-3)/2}e^{-\sigma|\mathrm{Im}\,z|} \le C\sigma^{-2\nu}\langle\sigma\rangle^{-5/2}|\mathrm{Im}\,z|^{-(n+2)/2}. \tag{2.16}$$

By (2.16), the norm in the LHS of (2.14) is upper bounded by

$$C\int_0^\infty \sigma^{n-1}\left|R^{\pm}_{h}(\sigma, z)\right| d\sigma = C\int_0^\infty \sigma^{n-1}\left|R^{\pm}_{1}(\sigma, z)\right| d\sigma$$

$$\le C|\mathrm{Im}\,z|^{-(n+2)/2}\int_0^\infty \langle\sigma\rangle^{-3/2}d\sigma \le C|\mathrm{Im}\,z|^{-(n+2)/2}.$$

To prove (2.13) and (2.15), we will use the identity

$$\mathcal{R}^{\pm}_{h}(z)\left(1 + h^2V\mathcal{R}^{\pm}_{0,h}(z)\right) = \mathcal{R}^{\pm}_{0,h}(z), \quad \pm\mathrm{Im}\,z > 0. \tag{2.17}$$

Observe that $1 + h^2V\mathcal{R}^{\pm}_{0,h}(0) = 1 - V\Delta^{-1}$, which is supposed to be invertible on L^1 with a bounded inverse denoted by T. Thus, it follows from (2.12) that there exists a constant $h_0 > 0$ so that for $h \ge h_0$ the operator $1 + h^2V\mathcal{R}^{\pm}_{0,h}(z)$ is invertible on L^1 with an inverse satisfying

$$\left\|\left(1 + h^2V\mathcal{R}^{\pm}_{0,h}(z)\right)^{-1}\right\|_{L^1\to L^1} \le C, \quad z \in \mathbf{C}^{\pm}_{\varphi}, \tag{2.18}$$

with a constant $C>0$ independent of z and h. Hence, we can write

$$\mathcal{R}_h^{\pm}(z)=\mathcal{R}_{0,h}^{\pm}(z)\left(1+h^2V\mathcal{R}_{0,h}^{\pm}(z)\right)^{-1}. \tag{2.19}$$

Now (2.13) follows from (2.11), (2.18) and (2.19), while (2.15) follows from (2.14), (2.18) and (2.19). □

Clearly, (2.1) and (2.2) follow from (2.5) and (2.14), (2.15), respectively. To prove (2.3) we rewrite the identity (2.19) in the form

$$\mathcal{R}_h^{\pm}(z)-\mathcal{R}_{0,h}^{\pm}(z)T$$

$$=\mathcal{R}_{0,h}^{\pm}(z)T\left(h^2V\mathcal{R}_{0,h}^{\pm}(z)-h^2V\mathcal{R}_{0,h}^{\pm}(0)\right)T\left(1+\left(h^2V\mathcal{R}_{0,h}^{\pm}(z)-h^2V\mathcal{R}_{0,h}^{\pm}(0)\right)T\right)^{-1}. \tag{2.20}$$

By Lemma 2.2, (2.18) and (2.20) we conclude

$$\left\|\mathcal{R}_h^{\pm}(z)-\mathcal{R}_{0,h}^{\pm}(z)T\right\|_{L^1\to L^1}\le Ch^{-\beta}|\operatorname{Im}z|^{-q},\quad h\ge h_0,\ z\in\mathbf{C}_\varphi^{\pm},\ \operatorname{Im}z\neq 0, \tag{2.21}$$

with constants $C,q,\beta>0$ independent of z and h. Now (2.3) follows from (2.5) and (2.21). □

Let $\psi_1\in C_0^\infty((0,+\infty))$, $\psi_1=1$ on $\operatorname{supp}\psi$.

Proposition 2.3 *Under the assumptions of Theorem 1.1, there exist positive constants h_0 and β so that we have the estimates*

$$\int_{-\infty}^{\infty}\left\|Ve^{itG_0}\psi(h^2G_0)\right\|_{L^1\to L^1}dt\le Ch^{-\beta},\quad h\ge 1, \tag{2.22}$$

$$\int_{-\infty}^{\infty}\left\|V\psi(h^2G)e^{itG_0}\psi_1(h^2G_0)\right\|_{L^1\to L^1}dt\le Ch^{-\beta},\quad h\ge h_0. \tag{2.23}$$

Proof. It is shown in [7] (Section 2) that the kernel of the operator $e^{itG_0}\psi(h^2G_0)$ is of the form $K_h(|x-y|,t)$ with a function K_h satisfying

$$K_h(\sigma,t)=h^{-n}K_1(\sigma h^{-1},th^{-2}),$$

$$|K_1(\sigma,t)|\le C|t|^{-s-1/2}\sigma^{s-(n-1)/2},\quad 0\le s\le (n-1)/2,\ \sigma>0,\ t\neq 0.$$

Hence, for all $0\le s\le (n-1)/2$, $\sigma>0$, $t\neq 0$, $h>0$, we have

$$|K_h(\sigma,t)|\le Ch^{s-(n-1)/2}|t|^{-s-1/2}\sigma^{s-(n-1)/2},$$

which together with (1.4) imply

$$\left\|Ve^{itG_0}\psi(h^2G_0)\right\|_{L^1\to L^1}\le Ch^{s-(n-1)/2}|t|^{-s-1/2},\quad 1/2-\epsilon\le s\le 1/2+\epsilon, \tag{2.24}$$

where $0<\epsilon\ll 1$. Clearly, (2.22) follows from (2.24). Furthermore, using (2.5), (2.13), (2.14) and (2.24), we get

$$\left\|V\left(\psi(h^2G)-\psi(h^2G_0)\right)e^{itG_0}\psi_1(h^2G_0)\right\|_{L^1\to L^1}$$

$$\le Ch^2\sum_{\pm}\int_{\mathbf{C}_\varphi^{\pm}}\left|\frac{\partial\widetilde{\varphi}}{\partial\bar z}(z)\right|\left\|V\mathcal{R}_h^{\pm}(z)Ve^{itG_0}\psi_1(h^2G_0)\mathcal{R}_{0,h}^{\pm}(z)\right\|_{L^1\to L^1}L(dz)$$

$$\leq Ch^2 \sum_{\pm} \int_{\mathbf{C}_\varphi^\pm} \left| \frac{\partial \widetilde{\varphi}}{\partial \bar{z}}(z) \right| \left\| V \mathcal{R}_h^\pm(z) \right\|_{L^1 \to L^1} \left\| V e^{itG_0} \psi_1(h^2 G_0) \right\|_{L^1 \to L^1} \left\| \mathcal{R}_{0,h}^\pm(z) \right\|_{L^1 \to L^1} L(dz)$$

$$\leq Ch^{s-(n-1)/2} |t|^{-s-1/2} \sum_{\pm} \int_{\mathbf{C}_\varphi^\pm} \left| \frac{\partial \widetilde{\varphi}}{\partial \bar{z}}(z) \right| |\mathrm{Im}\, z|^{-q} L(dz)$$

$$\leq Ch^{s-(n-1)/2} |t|^{-s-1/2}, \quad 1/2 - \epsilon \leq s \leq 1/2 + \epsilon, \tag{2.25}$$

which clearly implies (2.23). □

3 Proof of Theorem 1.1

Denote

$$\Psi(t,h) = e^{itG} \psi(h^2 G) - T^* e^{itG_0} \psi(h^2 G_0) T,$$

T being given by (2.4). We will first show that (1.5) follows from the following

Proposition 3.1 *Under the assumptions of Theorem 1.1, there exist positive constants C, h_0 and β so that for $h \geq h_0$ we have*

$$\|\Psi(t,h)\|_{L^1 \to L^\infty} \leq Ch^{-\beta} |t|^{-n/2}, \quad t \neq 0. \tag{3.1}$$

Recall that $\chi_a(\sigma) = \chi_1(\sigma/a)$, $a > 0$ small. Then we can write the function η_a as follows

$$\eta_a(\sigma) = \int_{a^{-1}}^{\infty} \psi(\sigma\theta) \frac{d\theta}{\theta}, \quad \sigma > 0,$$

where $\psi(\sigma) = \sigma \chi_1'(\sigma) \in C_0^\infty((0,+\infty))$. Thus, we obtain from (3.1),

$$\left\| e^{itG} \eta_a(G) - T^* e^{itG_0} \eta_a(G_0) T \right\|_{L^1 \to L^\infty} \leq \int_{a^{-1}}^{\infty} \left\| \Psi(t, \sqrt{\theta}) \right\|_{L^1 \to L^\infty} \frac{d\theta}{\theta}$$

$$\leq C|t|^{-n/2} \int_{a^{-1}}^{\infty} \theta^{-1-\beta/2} d\theta \leq C|t|^{-n/2}, \tag{3.2}$$

provided a is taken small enough. Clearly, (1.5) follows from (3.2).

Proof of Proposition 3.1. We will first prove the following

Proposition 3.2 *Under the assumptions of Theorem 1.1, there exist positive constants C, h_0 and β so that for $h \geq h_0$ we have*

$$\int_{-\infty}^{\infty} \left\| V e^{itG} \psi(h^2 G) \right\|_{L^1 \to L^1} dt \leq Ch^{-\beta}. \tag{3.3}$$

Proof. Using Duhamel's formula

$$e^{itG} = e^{itG_0} + i \int_0^t e^{i(t-\tau)G} V e^{i\tau G_0} d\tau,$$

we get the identity

$$e^{itG} \psi(h^2 G) = \psi(h^2 G) e^{itG_0} \psi_1(h^2 G_0) T + e^{itG} \psi(h^2 G) \left(\psi_1(h^2 G) - \psi_1(h^2 G_0) T \right)$$

$$+ i \int_0^t \psi(h^2 G) e^{i(t-\tau)G} V e^{i\tau G_0} \psi_1(h^2 G_0) T d\tau. \tag{3.4}$$

Using Propositions 2.1 and 2.3, (3.4) together with Young's inequality we obtain

$$\int_{-\infty}^{\infty}\left\|Ve^{itG}\psi(h^2G)\right\|_{L^1\to L^1}dt \le Ch^{-\beta}+Ch^{-\beta}\int_{-\infty}^{\infty}\left\|Ve^{itG}\psi(h^2G)\right\|_{L^1\to L^1}dt$$

$$+\int_{-\infty}^{\infty}\int_0^t\left\|V\psi(h^2G)e^{i(t-\tau)G}\right\|_{L^1\to L^1}\left\|Ve^{i\tau G_0}\psi_1(h^2G_0)\right\|_{L^1\to L^1}d\tau dt$$

$$\le Ch^{-\beta}+Ch^{-\beta}\int_{-\infty}^{\infty}\left\|Ve^{itG}\psi(h^2G)\right\|_{L^1\to L^1}dt,$$

which clearly implies (3.3) if we take h large enough. □

Using Duhamel's formula

$$e^{itG}=e^{itG_0}+i\int_0^t e^{i(t-\tau)G_0}Ve^{i\tau G}d\tau,$$

we get the identity

$$\Psi(t;h)=\sum_{j=1}^2\Psi_j(t;h), \tag{3.5}$$

where

$$\Psi_1(t;h)=T^*\psi_1(h^2G_0)e^{itG_0}\left(\psi(h^2G)-\psi(h^2G_0)T\right)+\left(\psi_1(h^2G)-T^*\psi_1(h^2G_0)\right)e^{itG}\psi(h^2G),$$

$$\Psi_2(t;h)=i\int_0^t T^*\psi_1(h^2G_0)e^{i(t-\tau)G_0}Ve^{i\tau G}\psi(h^2G)d\tau.$$

By (2.1) and (2.3) together with the well known estimate

$$\left\|e^{itG_0}\right\|_{L^1\to L^\infty}\le C|t|^{-n/2},$$

we get

$$\|\Psi_1(t;h)f\|_{L^\infty}\le Ch^{-\beta}|t|^{-n/2}\|f\|_{L^1}+Ch^{-\beta}\|\Psi(t;h)f\|_{L^\infty}, \quad t\neq 0. \tag{3.6}$$

By Propositions 2.3 and 3.2, $\forall f\in L^1$, $t>0$, we have

$$t^{n/2}\|\Psi_2(t;h)f\|_{L^\infty}$$

$$\le C\int_0^{t/2}(t-\tau)^{n/2}\left\|\psi_1(h^2G_0)e^{i(t-\tau)G_0}\right\|_{L^1\to L^\infty}\left\|Ve^{i\tau G}\psi(h^2G)f\right\|_{L^1}d\tau$$

$$+C\int_{t/2}^t\left\|\psi_1(h^2G_0)e^{i(t-\tau)G_0}V\right\|_{L^\infty\to L^\infty}\tau^{n/2}\left\|e^{i\tau G}\psi(h^2G)f\right\|_{L^\infty}d\tau$$

$$\le C\int_{-\infty}^{\infty}\left\|Ve^{i\tau G}\psi(h^2G)f\right\|_{L^1}d\tau$$

$$+C\sup_{t/2\le\tau\le t}\tau^{n/2}\left\|e^{i\tau G}\psi(h^2G)f\right\|_{L^\infty}\int_{-\infty}^{\infty}\left\|Ve^{i\tau G_0}\psi_1(h^2G_0)\right\|_{L^1\to L^1}d\tau$$

$$\le Ch^{-\beta}\|f\|_{L^1}+Ch^{-\beta}\sup_{t/2\le\tau\le t}\tau^{n/2}\left\|e^{i\tau G}\psi(h^2G)f\right\|_{L^\infty}. \tag{3.7}$$

Combining (3.5), (3.6) and (3.7), we conclude, $\forall f\in L^1$, $t>0$,

$$t^{n/2}\|\Psi(t;h)f\|_{L^\infty}\le Ch^{-\beta}\|f\|_{L^1}+Ch^{-\beta}t^{n/2}\|\Psi(t;h)f\|_{L^\infty}$$

$$+Ch^{-\beta}\sup_{t/2\le\tau\le t}\tau^{n/2}\|\Psi(\tau;h)f\|_{L^\infty}. \tag{3.8}$$

Taking h big enough we can absorb the second and the third terms in the RHS of (3.8), thus obtaining (3.1). Clearly, the case of $t<0$ can be treated in the same way. □

A Appendix 1

We will prove the following

Lemma A.1 *Let $\psi \in C_0^\infty((0,+\infty))$. Then, for all $h>0$, $s\geq 0$, we have the estimates*

$$\left\|\psi(h^2G_0)\right\|_{L^1\to L^1} \leq C, \tag{A.1}$$

$$\left\|\langle x\rangle^s\psi(h^2G_0)\langle x\rangle^{-s}\right\|_{L^1\to L^2} \leq Ch^{-n/2}\langle h\rangle^s, \tag{A.2}$$

where the constant C is of the form

$$C = C'\sup_{0\leq k\leq k_0}\sup_{\lambda\in\mathbf{R}}|\partial_\lambda^k\psi(\lambda)|, \tag{A.3}$$

with some integer k_0 independent of ψ and a constant $C'>0$ depending on the support of ψ, only. Furthermore, if V satisfies (1.1) with $\delta>n/2$, we have the estimates (for $0<h\leq 1$)

$$\left\|\psi(h^2G)-\psi(h^2G_0)\right\|_{L^1\to L^1} \leq Ch^2, \tag{A.4}$$

$$\left\|\langle x\rangle^\delta\left(\psi(h^2G)-\psi(h^2G_0)\right)\right\|_{L^1\to L^2} \leq Ch^{2-n/2}. \tag{A.5}$$

Proof. The estimate (A.1) is proved in Section 2 using the formula (2.5) and (2.14). It can be also seen by using the fact that the kernel of the operator $\psi(h^2G_0)$ is of the form $k_h(|x-y|)$ with a function k_h satisfying

$$k_h(\sigma) = h^{-n}k_1(\sigma/h), \tag{A.6}$$

$$|k_1(\sigma)| \leq C_m\langle\sigma\rangle^{-m}, \quad \forall\sigma>0, \tag{A.7}$$

for all integers $m\geq 0$, with a constant C_m of the form

$$C_m = C'_m\sup_{0\leq j\leq j_m}\sup_{\lambda\in\mathbf{R}}|\partial_\lambda^j\psi(\lambda)|, \tag{A.8}$$

where j_m is some integer independent of ψ, while $C'_m>0$ depends on the support of ψ. By Young's inequality, the norm in the LHS of (A.1) is upper bounded by

$$\int_{\mathbf{R}^n}|k_h(|\xi|)|d\xi = \int_{\mathbf{R}^n}|k_1(|\xi|)|d\xi \leq C_{n+1}.$$

The norm in the LHS of (A.2) is upper bounded by

$$\sup_{y\in\mathbf{R}^n}\left(\int_{\mathbf{R}^n}\langle x\rangle^{2s}\langle y\rangle^{-2s}|k_h(|x-y|)|^2dx\right)^{1/2} \leq \left(\int_{\mathbf{R}^n}\langle x-y\rangle^{2s}|k_h(|x-y|)|^2dx\right)^{1/2}$$

$$\leq C\langle h\rangle^s\left(\int_{\mathbf{R}^n}\langle\xi/h\rangle^{2s}|k_h(|\xi|)|^2d\xi\right)^{1/2}$$

$$= C\langle h\rangle^s h^{-n/2}\left(\int_{\mathbf{R}^n}\langle\xi\rangle^{2s}|k_1(|\xi|)|^2d\xi\right)^{1/2} \leq C_{s_n}\langle h\rangle^s h^{-n/2},$$

where s_n is some integer depending on n and s. To prove (A.4) observe that by (2.5) we have

$$\psi(h^2G)-\psi(h^2G_0) = \frac{2h^2}{\pi}\int_{\mathbf{C}}\frac{\partial\widetilde{\varphi}}{\partial\bar z}(z)(h^2G_0-z^2)^{-1}V(h^2G-z^2)^{-1}zL(dz). \tag{A.9}$$

Clearly, (A.4) would follow from (A.9), (2.14) and the estimate (for $z \in \operatorname{supp} \widetilde{\varphi}$)

$$\left\| (h^2 G - z^2)^{-1} \right\|_{L^1 \to L^1} \le C |\operatorname{Im} z|^{-q}, \quad 0 < h \le 1, \operatorname{Im} z \ne 0. \tag{A.10}$$

Let $\phi \in C_0^\infty([1,2])$ be such that $\int \phi(\theta^2)\theta^{-1} d\theta = 1$. Given a parameter $0 < \varepsilon \ll 1$, we decompose the free resolvent as follows

$$(h^2 G_0 - z^2)^{-1} = \mathcal{A}_\varepsilon(z;h) + \mathcal{B}_\varepsilon(z;h), \tag{A.11}$$

where

$$\mathcal{A}_\varepsilon(z;h) = \int_0^1 f\left((\varepsilon\theta h)^2 G_0; (\varepsilon\theta)^2; z\right) \frac{d\theta}{\theta},$$

$$\mathcal{B}_\varepsilon(z;h) = \int_1^\infty f\left((\varepsilon\theta h)^2 G_0; (\varepsilon\theta)^2; z\right) \frac{d\theta}{\theta},$$

where

$$f(\lambda; \mu; z) = \frac{\phi(\lambda)}{\lambda\mu^{-1} - z^2}.$$

It is easy to see that there exist constants $0 < \mu_1 < \mu_2$ so that the function f satisfies the following bounds

$$\left|\partial_\lambda^j f(\lambda;\mu;z)\right| \le C_j \mu, \quad 0 < \mu \le \mu_1, \tag{A.12}$$

$$\left|\partial_\lambda^j f(\lambda;\mu;z)\right| \le C_j' |\operatorname{Im} z|^{-j-1}, \quad \mu_1 \le \mu \le \mu_2, \tag{A.13}$$

$$\left|\partial_\lambda^j f(\lambda;\mu;z)\right| \le C_j'', \quad \mu \ge \mu_2, \tag{A.14}$$

for every integer $j \ge 0$. By (A.1), (A.3) and (A.12), we have

$$\left\| f\left((\varepsilon\theta h)^2 G_0; (\varepsilon\theta)^2; z\right) \right\|_{L^1 \to L^1} \le C(\varepsilon\theta)^2, \quad 0 < \theta \le 1, \tag{A.15}$$

provided $\varepsilon > 0$ is taken small enough. We deduce from (A.15),

$$\|\mathcal{A}_\varepsilon(z;h)\|_{L^1 \to L^1} \le C\varepsilon^2, \quad z \in \operatorname{supp} \widetilde{\varphi}, \tag{A.16}$$

with a constant $C > 0$ independent of z, h and ε. By (A.2), (A.3), (A.12)-(A.14), we have

$$\left\| \langle x\rangle^s f\left((\varepsilon\theta h)^2 G_0; (\varepsilon\theta)^2; z\right) \langle x\rangle^{-s} \right\|_{L^1 \to L^2} \le C(\varepsilon\theta h)^{-n/2} \langle \varepsilon\theta h\rangle^s |\operatorname{Im} z|^{-q}, \tag{A.17}$$

with constants $C, q > 0$ independent of z, θ, h and ε. We deduce from (A.17),

$$\begin{aligned} \|\mathcal{B}_\varepsilon(z;h)\|_{L^1 \to L^2} &\le C_\varepsilon' h^{-n/2} |\operatorname{Im} z|^{-q} \int_1^\infty \theta^{-1-n/2} d\theta \\ &\le C_\varepsilon h^{-n/2} |\operatorname{Im} z|^{-q}, \quad z \in \operatorname{supp} \widetilde{\varphi}, \end{aligned} \tag{A.18}$$

with a constant $C_\varepsilon > 0$ independent of z and h. It follows from (A.16) that the operator $1 + h^2 V \mathcal{A}_\varepsilon(z;h)$ is invertible on L^1, provided $\varepsilon > 0$ is taken small enough, independent of h. Therefore, we can write the identity

$$\left(h^2 G - z^2\right)^{-1} = \left(h^2 G_0 - z^2\right)^{-1} + h^2 \left(h^2 G - z^2\right)^{-1} V \left(h^2 G_0 - z^2\right)^{-1}, \tag{A.19}$$

in the form

$$\left(h^2 G - z^2\right)^{-1} = \mathcal{M}(z;h) + h^2 \left(h^2 G - z^2\right)^{-1} \mathcal{N}(z;h), \tag{A.20}$$

where the operators

$$\mathcal{M}(z;h) = \left(h^2 G_0 - z^2\right)^{-1} \left(1 + h^2 V \mathcal{A}_\varepsilon(z;h)\right)^{-1},$$

$$\mathcal{N}(z;h) = V\mathcal{B}_\varepsilon(z;h)\left(1 + h^2 V\mathcal{A}_\varepsilon(z;h)\right)^{-1},$$

satisfy the estimates

$$\|\mathcal{M}(z;h)\|_{L^1\to L^1} + \|\mathcal{N}(z;h)\|_{L^1\to L^1} \le C|\mathrm{Im}\, z|^{-q}, \tag{A.21}$$

$$\|\mathcal{N}(z;h)\|_{L^1\to L^2} \le Ch^{-n/2}|\mathrm{Im}\, z|^{-q}. \tag{A.22}$$

By (A.20) we have

$$\begin{aligned}\left(h^2G - z^2\right)^{-1} &= \sum_{j=0}^{J-1} \mathcal{M}(z;h)\mathcal{N}(z;h)^j + h^{2J}\left(h^2G - z^2\right)^{-1}\mathcal{N}(z;h)^J \\ &= \sum_{j=0}^{J-1} \mathcal{M}(z;h)\mathcal{N}(z;h)^j + h^{2J}\left(h^2G_0 - z^2\right)^{-1}\mathcal{N}(z;h)^J \\ &\quad + h^{2J+2}\left(h^2G_0 - z^2\right)^{-1} V\left(h^2G - z^2\right)^{-1}\mathcal{N}(z;h)^J,\end{aligned} \tag{A.23}$$

for every integer $J \ge 1$. By (A.22) and (2.14), we obtain

$$\begin{aligned}&\left\|\left(h^2G_0 - z^2\right)^{-1} V\left(h^2G - z^2\right)^{-1}\mathcal{N}(z;h)\right\|_{L^1\to L^1} \\ &\le \|V\|_{L^2}\left\|\left(h^2G_0 - z^2\right)^{-1}\right\|_{L^1\to L^1}\left\|\left(h^2G - z^2\right)^{-1}\right\|_{L^2\to L^2}\|\mathcal{N}(z;h)\|_{L^1\to L^2} \\ &\le Ch^{-n/2}|\mathrm{Im}\, z|^{-q_2}.\end{aligned} \tag{A.24}$$

Now, (A.10) follows from (A.21), (A.23) and (A.24).

To prove (A.5) we rewrite (A.20) in the form

$$\left(h^2G - z^2\right)^{-1} - \left(h^2G_0 - z^2\right)^{-1} = \sum_{j=1}^{3} \mathcal{F}_j(z;h), \tag{A.25}$$

where

$$\mathcal{F}_1(z;h) = h^2\mathcal{A}_\varepsilon(z;h)V\mathcal{A}_\varepsilon(z;h)\left(1 + h^2V\mathcal{A}_\varepsilon(z;h)\right)^{-1},$$

$$\mathcal{F}_2(z;h) = h^2\mathcal{B}_\varepsilon(z;h)V\mathcal{A}_\varepsilon(z;h)\left(1 + h^2V\mathcal{A}_\varepsilon(z;h)\right)^{-1},$$

$$\mathcal{F}_3(z;h) = h^2\left(h^2G - z^2\right)^{-1} V\mathcal{B}_\varepsilon(z;h)\left(1 + h^2V\mathcal{A}_\varepsilon(z;h)\right)^{-1}.$$

It is easy to see that we have the estimate

$$\left\|\langle x\rangle^s\left(h^2G - z^2\right)^{-1}\langle x\rangle^{-s}\right\|_{L^2\to L^2} \le C|\mathrm{Im}\, z|^{-q}, \quad z \in \mathrm{supp}\,\widetilde{\varphi},\ 0 < h \le 1, \tag{A.26}$$

for every $s \ge 0$ with constants $C, q > 0$ depending on s but independent of z and h. By (A.18) and (A.26),

$$\left\|\langle x\rangle^\delta\mathcal{F}_3(z;h)\right\|_{L^1\to L^2} \le Ch^{2-n/2}|\mathrm{Im}\, z|^{-q}, \quad z \in \mathrm{supp}\,\widetilde{\varphi},\ 0 < h \le 1. \tag{A.27}$$

Observe now that we can write the operator $\mathcal{A}_\varepsilon(z;h)$ in the form

$$\mathcal{A}_\varepsilon(z;h) = \chi_\varepsilon^{(3)}(h^2G_0)\left(h^2G_0 - z^2\right)^{-1},$$

where

$$\chi_\varepsilon^{(3)}(\sigma) = \int_0^{\varepsilon\sigma^{1/2}} \phi(\theta^2)\frac{d\theta}{\theta}.$$

Similarly, we can decompose the operator $\mathcal{B}_\varepsilon(z;h)$ as $\mathcal{B}_\varepsilon^{(1)}+\mathcal{B}_\varepsilon^{(2)}$, where

$$\mathcal{B}_\varepsilon^{(j)}(z;h)=\chi_\varepsilon^{(j)}(h^2G_0)\left(h^2G_0-z^2\right)^{-1},\quad j=1,2,$$

$$\chi_\varepsilon^{(1)}(\sigma)=\int_{\varepsilon\sigma^{1/2}}^{\varepsilon^{-1}\sigma^{1/2}}\phi(\theta^2)\frac{d\theta}{\theta},\quad \chi_\varepsilon^{(2)}(\sigma)=\int_{\varepsilon^{-1}\sigma^{1/2}}^{\infty}\phi(\theta^2)\frac{d\theta}{\theta}.$$

Taking $\varepsilon>0$ small enough we can arrange that $\operatorname{supp}\chi_\varepsilon^{(j)}\cap\operatorname{supp}\varphi=\emptyset$, $j=2,3$, so the operator-valued functions $\mathcal{A}_\varepsilon(z;h)$ and $\mathcal{B}_\varepsilon^{(2)}(z;h)$ are analytic on $\operatorname{supp}\widetilde{\varphi}$. Therefore, we can write (A.9) in the form

$$\psi(h^2G)-\psi(h^2G_0)=\frac{2}{\pi}\sum_{j=3}^{4}\int_{\mathbf{C}}\frac{\partial\widetilde{\varphi}}{\partial\bar z}(z)\mathcal{F}_j(z;h)zL(dz),\tag{A.28}$$

where

$$\mathcal{F}_4(z;h)=h^2\mathcal{B}_\varepsilon^{(1)}(z;h)V\mathcal{A}_\varepsilon(z;h)\left(1+h^2V\mathcal{A}_\varepsilon(z;h)\right)^{-1}.$$

By (A.17) (with $s=\delta$), we have

$$\begin{aligned}\left\|\langle x\rangle^\delta\mathcal{F}_4(z;h)\right\|_{L^1\to L^2}&\le Ch^2\left\|\langle x\rangle^\delta\mathcal{B}_\varepsilon^{(1)}(z;h)\langle x\rangle^{-\delta}\right\|_{L^1\to L^2}\\ &\le Ch^{2-n/2}|\operatorname{Im}z|^{-q},\quad z\in\operatorname{supp}\widetilde{\varphi},\ 0<h\le 1.\end{aligned}\tag{A.29}$$

Now (A.5) follows from (A.27)-(A.29). $\Box$

B Appendix 2

Combining some ideas from [6],[7] and [4] we will prove the following

Theorem B.1 *Let $n\ge 4$, let V satisfy (1.1) with $\delta>n-1$ as well as (1.3). Then, for every $a>0$ we have the estimate*

$$\left\|e^{itG}\chi_a(G)\right\|_{L^1\to L^\infty}\le C|t|^{-n/2},\quad t\neq 0.\tag{B.1}$$

Remark. Note that (B.1) is proved in [4] for potentials satisfying (1.1) with $\delta>n$, the condition (1.3) as well as an extra technical assumption. Here we eliminate this extra assumption.

Proof. The key point in the proof in [4] is the bound

$$\left\|e^{-itG_0}Ve^{itG_0}\right\|_{L^1\to L^1}\le\|\widehat{V}\|_{L^1},\quad\forall t.\tag{B.2}$$

Combining (B.2) with Duhamel's formula one easily gets

$$\left\|e^{-itG_0}Ve^{itG}\right\|_{L^1\to L^1}\le C,\quad |t|\le 1,\tag{B.3}$$

with a constant $C>0$ independent of t. In what follows we will derive (B.1) from (B.2) and (B.3). To this end, given a function $\psi\in C_0^\infty((0,+\infty))$ and a parameter $0<h\le 1$, as in [6], [7], denote

$$\Psi(t,h)=e^{itG}\psi(h^2G)-e^{itG_0}\psi(h^2G_0),$$

$$F(t)=i\int_0^t e^{i(t-\tau)G_0}Ve^{i\tau G_0}d\tau.$$

As in these papers, it is easy to see that (B.1) follows from the following

Theorem B.2 *Under the assumptions of Theorem B.1, there exist constants $C, \beta > 0$ so that we have the estimates (for $0 < h \leq 1$, $t \neq 0$)*

$$\|F(t)\|_{L^1 \to L^\infty} \leq C|t|^{-n/2}, \tag{B.4}$$

$$\left\|\Psi(t,h) - F(t)\psi(h^2 G_0)\right\|_{L^1 \to L^\infty} \leq C h^\beta |t|^{-n/2}. \tag{B.5}$$

Proof. Clearly, (B.4) follows from (B.2) for $|t| \leq 2$. Let $|t| \geq 2$. Without loss of generality we may suppose $t \geq 2$. Write $F = F_1 + F_2$, where

$$F_1(t) = i \int_1^{t-1} e^{i(t-\tau)G_0} V e^{i\tau G_0} d\tau,$$

$$F_2(t) = i \left(\int_0^1 + \int_{t-1}^t \right) e^{i(t-\tau)G_0} V e^{i\tau G_0} d\tau.$$

It follows from (B.2) that $F_2(t)$ satisfies (B.4). To deal with the operator $F_1(t)$, observe that its kernel is of the form

$$c_n \int_{\mathbf{R}^n} U(|x-\xi|^2/4, |y-\xi|^2/4, t) V(\xi) d\xi,$$

where c_n is a constant and

$$U(\sigma_1, \sigma_2, t) = \int_1^{t-1} e^{i\sigma_1/(t-\tau) + i\sigma_2/\tau} (t-\tau)^{-n/2} \tau^{-n/2} d\tau.$$

To prove that $F_1(t)$ satisfies (B.4), it suffices to show that

$$|U(\sigma_1, \sigma_2, t)| \leq C t^{-n/2} \left(\sigma_1^{-1/2} + \sigma_2^{-1/2} \right), \quad \forall \sigma_1, \sigma_2 > 0,\ t \geq 2. \tag{B.6}$$

To do so, observe that

$$U(\sigma_1, \sigma_2, t) = t^{-n+1} \left(u(\sigma_1 t^{-1}, \sigma_2 t^{-1}, t^{-1}) + u(\sigma_2 t^{-1}, \sigma_1 t^{-1}, t^{-1}) \right), \tag{B.7}$$

where

$$u(\sigma_1', \sigma_2', \kappa) = \int_\kappa^{1/2} e^{i\sigma_1'/(1-\tau') + i\sigma_2'/\tau'} (1-\tau')^{-n/2} (\tau')^{-n/2} d\tau'.$$

It is easy to see that (B.6) follows from (B.7) and the bound

$$|u(\sigma_1', \sigma_2', \kappa)| \leq C \kappa^{-(n-3)/2} (\sigma_2')^{-1/2}, \quad \forall \sigma_1', \sigma_2' > 0,\ 0 < \kappa \leq 1/2. \tag{B.8}$$

To prove (B.8), we make a change of variables $\mu = 1/\tau'$ and write the function u in the form

$$u(\sigma_1', \sigma_2', \kappa) = \int_2^{\kappa^{-1}} e^{i\varphi(\mu, \sigma_1', \sigma_2')} \left(\frac{\mu}{\mu - 1} \right)^{n/2} \mu^{n/2-2} d\mu,$$

where

$$\varphi(\mu, \sigma_1', \sigma_2') = \mu\sigma_2' + \frac{\mu}{\mu-1}\sigma_1'.$$

We have

$$|u(\sigma_1', \sigma_2', \kappa)| \leq C \int_2^{\kappa^{-1}} \mu^{n/2-2} d\mu \leq C\kappa^{-(n-2)/2}. \tag{B.9}$$

Furthermore, observe that

$$\varphi'(\mu) = \frac{d\varphi}{d\mu} = \sigma_2' - \frac{\sigma_1'}{(\mu-1)^2},$$

so φ' vanishes at $\mu_0 = 1 + (\sigma_1'/\sigma_2')^{1/2}$. We will consider now two cases.

Case 1. $\mu_0 \notin [3/2, 3\kappa^{-1}/2]$. Then, we have

$$|\varphi'(\mu)| \geq \sigma_2' \frac{|\mu - \mu_0|}{\mu - 1} \geq \frac{\sigma_2'}{10}, \quad \mu \in [2, \kappa^{-1}].$$

Therefore, integrating by parts, we obtain

$$u(\sigma_1', \sigma_2', \kappa) = \int_2^{\kappa^{-1}} (i\varphi')^{-1} \left(\frac{\mu}{\mu - 1}\right)^{n/2} \mu^{n/2-2} de^{i\varphi}$$

$$= e^{i\varphi}(i\varphi')^{-1} \left(\frac{\mu}{\mu - 1}\right)^{n/2} \mu^{n/2-2} \Bigg|_2^{\kappa^{-1}} - \int_2^{\kappa^{-1}} e^{i\varphi} f(\mu) d\mu, \tag{B.10}$$

where

$$f(\mu) = \frac{d}{d\mu}\left((i\varphi')^{-1}\left(\frac{\mu}{\mu-1}\right)^{n/2}\mu^{n/2-2}\right)$$

$$= (i\varphi')^{-1}\frac{d}{d\mu}\left(\left(\frac{\mu}{\mu-1}\right)^{n/2}\mu^{n/2-2}\right) + \frac{i\varphi''}{\varphi'^2}\left(\frac{\mu}{\mu-1}\right)^{n/2}\mu^{n/2-2}.$$

Since

$$\left|\frac{\varphi''}{\varphi'}\right| \leq \frac{2\sigma_1'(\mu-1)^{-2}}{(\mu-1)\,|\sigma_2' - \sigma_1'(\mu-1)^{-2}|} \leq \frac{2}{\mu-1}\left(1 + \frac{\sigma_2'}{|\varphi'|}\right) \leq \frac{22}{\mu-1},$$

we have (for $\mu \geq 2$)

$$|f(\mu)| \leq C(\sigma_2')^{-1}\mu^{n/2-3}. \tag{B.11}$$

By (B.10) and (B.11),

$$|u(\sigma_1', \sigma_2', \kappa)| \leq C(\sigma_2')^{-1}\kappa^{-(n-4)/2}. \tag{B.12}$$

Clearly, in this case (B.8) follows from (B.9) and (B.12).

Case 2. $\mu_0 \in [3/2, 3\kappa^{-1}/2]$. Denote $I(\mu_0) = [9\mu_0/10, 11\mu_0/10] \cap [2, \kappa^{-1}]$. We write the function u as $u_1 + u_2$, where

$$u_1 = \int_{I(\mu_0)} e^{i\varphi}\left(\frac{\mu}{\mu-1}\right)^{n/2}\mu^{n/2-2}d\mu = \mu_0^{n/2-1}\int_{\widetilde{I}(\mu_0)} e^{i\lambda\phi(z)}g(z)dz, \tag{B.13}$$

where we have made a change of variables $\mu = \mu_0(1+z)$, $\widetilde{I}(\mu_0) \subset [-1/10, 1/10]$, $\lambda = \mu_0\sigma_2'$,

$$g(z) = \left(\frac{1+z}{1+z-\mu_0^{-1}}\right)^{n/2}(1+z)^{n/2-2},$$

$$\phi(z) = (1+z)\left(1 + \frac{(\mu_0-1)^2}{\mu_0(1+z)-1}\right) = \mu_0 + \frac{\mu_0}{\mu_0-1}z^2 + O(z^3), \quad |z| \ll 1,$$

uniformly in μ_0. It is easy to see that we have the estimate

$$\left|\int_0^a e^{i\lambda\phi(z)}g(z)dz\right| \leq C\lambda^{-1/2}, \quad |a| \leq 1/10. \tag{B.14}$$

Indeed, the functions $g(z)$ and $\phi(z)$ are analytic in $|z| \leq 1/10$ with $|g(z)|$ bounded there uniformly in μ_0. Therefore, we can change the contour of integration to obtain (with some $0 < \gamma \ll 1$)

$$\left|\int_0^a e^{i\lambda\phi(z)}g(z)dz\right| \leq \left|\int_0^a e^{i\lambda\phi(e^{i\gamma}y)}g(e^{i\gamma}y)dy\right| + \left|a\int_0^\gamma e^{i\lambda\phi(e^{i\theta}a)}g(e^{i\theta}a)d\theta\right|$$

$$\leq C_1 \int_0^a e^{-C\lambda y^2} dy + C_1' \int_0^\gamma e^{-C'\lambda\theta} d\theta = O(\lambda^{-1/2}),$$

with some constants $C, C', C_1, C_1' > 0$. By (B.13) and (B.14) we conclude

$$|u_1| \leq C(\sigma_2')^{-1/2} \mu_0^{(n-3)/2} \leq \widetilde{C}(\sigma_2')^{-1/2} \kappa^{-(n-3)/2}. \tag{B.15}$$

On the other hand, if $\mu \in [2, \kappa^{-1}] \setminus I(\mu_0)$, then

$$\frac{|\mu - \mu_0|}{\mu - 1} \geq C > 0,$$

so we can bound from below $|\varphi'(\mu)|$. Therefore, the function u_2 can be treated in the same way as does u in Case 1. Thus, u_2 satisfies (B.8) and hence, in view of (B.15), so does u. This completes the proof of (B.4).

It suffices to prove (B.5) for $0 < h \leq h_0$ with some constant $0 < h_0 \leq 1$, since for $h_0 \leq h \leq 1$ it follows from (B.4) and the estimate of the $L^1 \to L^\infty$ norm of $\Psi(t, h)$ proved in [7] for the larger class of potentials satisfying (1.1) with $\delta > (n+2)/2$ (without using (1.3)). Without loss of generality we may suppose $t > 0$. Now, using Duhamel's formula as in [6], [7] we get the identity

$$\Psi(t; h) - F(t)\psi(h^2 G_0) = \sum_{j=1}^{5} \Psi_j(t; h), \tag{B.16}$$

where

$$\Psi_1(t; h) = \psi_1(h^2 G_0) e^{itG_0} \left(\psi(h^2 G) - \psi(h^2 G_0)\right)$$
$$+ \left(\psi_1(h^2 G) - \psi_1(h^2 G_0)\right) e^{itG_0} \psi(h^2 G_0) + \left(\psi_1(h^2 G) - \psi_1(h^2 G_0)\right) \Psi(t; h),$$

$$\Psi_2(t; h) = i \left(\int_0^\gamma + \int_{t-\gamma}^t\right) \psi_1(h^2 G_0) e^{i(t-\tau)G_0} V e^{i\tau G} \psi(h^2 G) d\tau,$$

$$\Psi_3(t; h) = -i \left(\int_0^\gamma + \int_{t-\gamma}^t\right) e^{i(t-\tau)G_0} V e^{i\tau G_0} \psi(h^2 G_0) d\tau,$$

$$\Psi_4(t; h) = i \int_\gamma^{t-\gamma} \psi_1(h^2 G_0) e^{i(t-\tau)G_0} V \Psi(\tau; h) d\tau,$$

$$\Psi_5(t; h) = -i \int_\gamma^{t-\gamma} (1 - \psi_1)(h^2 G_0) e^{i(t-\tau)G_0} V e^{i\tau G_0} \psi(h^2 G_0) d\tau,$$

where $\psi_1 \in C_0^\infty((0, +\infty))$, $\psi_1 = 1$ on $\operatorname{supp}\psi$, and $0 < \gamma \ll 1$ is a parameter to be fixed later on, depending on h. In view of (A.4), we have

$$\|\Psi_1(t; h) f\|_{L^\infty} \leq C h^2 t^{-n/2} \|f\|_{L^1} + C h^2 \|\Psi(t; h) f\|_{L^\infty}, \quad \forall f \in L^1. \tag{B.17}$$

By (B.2) and (B.3),

$$\|\Psi_j(t; h) f\|_{L^\infty} \leq C\gamma t^{-n/2} \|f\|_{L^1} + C\gamma \|\Psi(t; h) f\|_{L^\infty}, \quad \forall f \in L^1,\ j = 2, 3, \tag{B.18}$$

with a constant $C > 0$ independent of t, h and γ.

Proposition B.3 *Let V satisfy (1.1) with $\delta > n - 1$. Then, there exist constants $C, \beta_1 > 0$ so that for $0 < h \leq 1$, $t \geq 2\gamma$, we have the estimate*

$$\|\Psi_4(t, h)\|_{L^1 \to L^\infty} \leq C h^{\beta_1} \gamma^{-(n-3)/2} t^{-n/2}. \tag{B.19}$$

Proof. We will make use of the following estimates proved in [7].

Proposition B.4 *Let V satisfy (1.1) with $\delta > (n+2)/2$. Then, for every $0 < \epsilon \ll 1$, $1/2 - \epsilon/4 \le s \le (n-1)/2$, $0 < h \le 1$, $t \neq 0$, we have the estimates*

$$\left\|\psi(h^2G_0)e^{itG_0}\langle x\rangle^{-s-1/2-\epsilon}\right\|_{L^2\to L^\infty} \le Ch^{s-(n-1)/2}|t|^{-s-1/2}, \qquad (B.20)$$

$$\left\|\Psi(t,h)\langle x\rangle^{-s-1/2-\epsilon}\right\|_{L^2\to L^\infty} \le Ch^{s-(n-3)/2-\epsilon/4}|t|^{-s-1/2}. \qquad (B.21)$$

By (B.20) and (B.21), we get (with some $0 < \varepsilon_0 \ll 1$)

$$\begin{aligned}
&\|\Psi_4(t,h)\|_{L^1\to L^\infty}\\
\le\ & C\int_\gamma^{t/2}\left\|\psi_1(h^2G_0)e^{i(t-\tau)G_0}\langle x\rangle^{-n/2-\varepsilon_0}\right\|_{L^2\to L^\infty}\left\|\langle x\rangle^{-(n-2)/2-\varepsilon_0}\Psi(\tau,h)\right\|_{L^1\to L^2}d\tau\\
+\ & C\int_{t/2}^{t-\gamma}\left\|\psi_1(h^2G_0)e^{i(t-\tau)G_0}\langle x\rangle^{-(n-2)/2-\varepsilon_0}\right\|_{L^2\to L^\infty}\left\|\langle x\rangle^{-n/2-\varepsilon_0}\Psi(\tau,h)\right\|_{L^1\to L^2}d\tau\\
\le\ & Ch^{\varepsilon_0/4}t^{-n/2}\int_\gamma^{t/2}\tau^{-(n-2)/2}d\tau + Ch^{\varepsilon_0/4}t^{-n/2}\int_{t/2}^{t-\gamma}(t-\tau)^{-(n-2)/2}d\tau\\
\le\ & Ch^{\varepsilon_0/4}\gamma^{-(n-3)/2}t^{-n/2}.
\end{aligned}$$

□

Proposition B.5 *Let V satisfy (1.1) with $\delta > n-1$. Then, for every $0 < \epsilon \ll 1$, $0 < h \le 1$, $t \ge 2\gamma$, we have the estimate*

$$\|\Psi_5(t,h)\|_{L^1\to L^\infty} \le C_\epsilon h^\epsilon\gamma^{-(n-3)/2-\epsilon}t^{-n/2}. \qquad (B.22)$$

Proof. We will make use of the fact that the kernel of the operator $e^{itG_0}\psi(h^2G_0)$ is of the form $K_h(|x-y|,t)$, where

$$K_h(\sigma,t) = \frac{\sigma^{-2\nu}}{(2\pi)^{\nu+1}}\int_0^\infty e^{it\lambda^2}\mathcal{J}_\nu(\sigma\lambda)\psi(h^2\lambda^2)\lambda d\lambda = h^{-n}K_1(\sigma h^{-1}, th^{-2}), \qquad (B.23)$$

where $\mathcal{J}_\nu(z) = z^\nu J_\nu(z)$, $J_\nu(z) = \left(H_\nu^+(z) + H_\nu^-(z)\right)/2$ is the Bessel function of order $\nu = (n-2)/2$. So, the kernel of the operator Ψ_5 is of the form

$$\int_{\mathbf{R}^n} W_h(|x-\xi|, |y-\xi|, t, \gamma)V(\xi)d\xi,$$

where

$$\begin{aligned}
W_h(\sigma_1,\sigma_2,t,\gamma) &= -i\int_\gamma^{t-\gamma}\widetilde{K}_h(\sigma_1, t-\tau)K_h(\sigma_2,\tau)d\tau\\
&= h^{-2n+2}W_1(\sigma_1h^{-1}, \sigma_2h^{-1}, th^{-2}, \gamma h^{-2}), \qquad (B.24)
\end{aligned}$$

where $\widetilde{K}_h$ is defined by replacing in the definition of K_h the function ψ by $1-\psi_1$. It is easy to see that (B.22) follows from the bound (for all σ_1, σ_2, $\gamma > 0$, $0 < \epsilon \ll 1$, $t \ge 2\gamma$)

$$|W_h(\sigma_1,\sigma_2,t,\gamma)| \le C_\epsilon h^\epsilon\gamma^{-(n-3)/2-\epsilon}t^{-n/2}\left(\sigma_1^{-n+2} + \sigma_1^{-1+\epsilon} + \sigma_2^{-n+2} + \sigma_2^{-1+\epsilon}\right). \qquad (B.25)$$

In view of (B.24), it suffices to prove (B.25) with $h = 1$. Now, observe that $W_1 = W_1^{(1)} - W_1^{(2)}$, where

$$W_1^{(1)}(\sigma_1,\sigma_2,t,\gamma) = \frac{(\sigma_1\sigma_2)^{-2\nu}}{4(2\pi)^{2\nu+2}}\int_0^\infty\int_0^\infty e^{i(t-\gamma)\lambda_1^2 + i\gamma\lambda_2^2}\rho(\lambda_1^2,\lambda_2^2)\mathcal{J}_\nu(\sigma_1\lambda_1)\mathcal{J}_\nu(\sigma_2\lambda_2)d\lambda_1^2d\lambda_2^2,$$

$$W_1^{(2)}(\sigma_1,\sigma_2,t,\gamma)=\frac{(\sigma_1\sigma_2)^{-2\nu}}{4(2\pi)^{2\nu+2}}\int_0^\infty\int_0^\infty e^{i(t-\gamma)\lambda_2^2+i\gamma\lambda_1^2}\rho(\lambda_1^2,\lambda_2^2)\mathcal{J}_\nu(\sigma_1\lambda_1)\mathcal{J}_\nu(\sigma_2\lambda_2)d\lambda_1^2d\lambda_2^2,$$

where the function

$$\rho(\lambda_1^2,\lambda_2^2)=\frac{(1-\psi_1)(\lambda_1^2)\psi(\lambda_2^2)}{\lambda_2^2-\lambda_1^2}=(1-\psi_1)(\lambda_1^2)\psi_1(\lambda_2^2)\frac{\psi(\lambda_2^2)-\psi(\lambda_1^2)}{\lambda_2^2-\lambda_1^2}$$

satisfies the bound

$$\left|\partial_{\lambda_1^2}^{\alpha_1}\partial_{\lambda_2^2}^{\alpha_2}\rho(\lambda_1^2,\lambda_2^2)\right|\le C_{\alpha_1,\alpha_2}\langle\lambda_1^2\rangle^{-1-\alpha_1},\quad\forall(\lambda_1,\lambda_2).\qquad (B.26)$$

Given any integers $0\le k,m<n/2$, since $\mathcal{J}_\nu(z)=O(z^{n-2})$ as $z\to 0$, we can integrate by parts to get

$$W_1^{(1)}(\sigma_1,\sigma_2,t,\gamma)=i^{-m-k}(t-\gamma)^{-k}\gamma^{-m}\frac{(\sigma_1\sigma_2)^{-2\nu}}{4(2\pi)^{2\nu+2}}\int_0^\infty\int_0^\infty e^{i(t-\gamma)\lambda_1^2+i\gamma\lambda_2^2}$$
$$\times\partial_{\lambda_1^2}^k\partial_{\lambda_2^2}^m\left(\rho(\lambda_1^2,\lambda_2^2)\mathcal{J}_\nu(\sigma_1\lambda_1)\mathcal{J}_\nu(\sigma_2\lambda_2)\right)d\lambda_1^2d\lambda_2^2,$$
$$W_1^{(2)}(\sigma_1,\sigma_2,t,\gamma)=i^{-m-k}(t-\gamma)^{-k}\gamma^{-m}\frac{(\sigma_1\sigma_2)^{-2\nu}}{4(2\pi)^{2\nu+2}}\int_0^\infty\int_0^\infty e^{i(t-\gamma)\lambda_2^2+i\gamma\lambda_1^2}$$
$$\times\partial_{\lambda_1^2}^m\partial_{\lambda_2^2}^k\left(\rho(\lambda_1^2,\lambda_2^2)\mathcal{J}_\nu(\sigma_1\lambda_1)\mathcal{J}_\nu(\sigma_2\lambda_2)\right)d\lambda_1^2d\lambda_2^2.$$

Using the inequality

$$\left|\int_{-\infty}^\infty e^{it\lambda^2}\varphi(\lambda)d\lambda\right|\le C|t|^{-1/2}\left\|\widehat{\varphi}\right\|_{L^1},\qquad (B.27)$$

we obtain (for $t\ge 2\gamma$)

$$\left|W_1^{(1)}(\sigma_1,\sigma_2,t,\gamma)\right|\le Ct^{-k-1/2}\gamma^{-m}(\sigma_1\sigma_2)^{-2\nu}\int_{-\infty}^\infty\left|\int_0^\infty\int_0^\infty e^{i\tau\lambda_1+i\gamma\lambda_2^2}\right.$$
$$\left.\times\lambda_1\partial_{\lambda_1^2}^k\partial_{\lambda_2^2}^m\left(\rho(\lambda_1^2,\lambda_2^2)\mathcal{J}_\nu(\sigma_1\lambda_1)\mathcal{J}_\nu(\sigma_2\lambda_2)\right)d\lambda_1d\lambda_2^2\right|d\tau,\qquad (B.28)$$
$$\left|W_1^{(2)}(\sigma_1,\sigma_2,t,\gamma)\right|\le Ct^{-k-1/2}\gamma^{-m}(\sigma_1\sigma_2)^{-2\nu}\int_{-\infty}^\infty\left|\int_0^\infty\int_0^\infty e^{i\tau\lambda_2+i\gamma\lambda_1^2}\right.$$
$$\left.\times\lambda_2\partial_{\lambda_1^2}^m\partial_{\lambda_2^2}^k\left(\rho(\lambda_1^2,\lambda_2^2)\mathcal{J}_\nu(\sigma_1\lambda_1)\mathcal{J}_\nu(\sigma_2\lambda_2)\right)d\lambda_2d\lambda_1^2\right|d\tau.\qquad (B.29)$$

Recall now that the function $\mathcal{J}_\nu$ is of the form $\mathcal{J}_\nu(z)=e^{iz}b_\nu^+(z)+e^{-iz}b_\nu^-(z)$, where $b_\nu^\pm(z)$ are symbols of order $(n-3)/2$ for $z\ge 1$, while near $z=0$ the function $\mathcal{J}_\nu(z)$ is equal to $z^{2\nu}$ times an analytic function. Therefore, it satisfies the bounds

$$\left|\partial_z^j\mathcal{J}_\nu(z)\right|\le Cz^{n-2-j}\langle z\rangle^{j-(n-1)/2},\quad\forall z>0,\,0\le j\le n-2,\qquad (B.30)$$

$$\left|\partial_z^j\mathcal{J}_\nu(z)\right|\le C_j\langle z\rangle^{(n-3)/2},\quad\forall z>0,\,j\ge 0.\qquad (B.31)$$

Moreover, the functions $b_\nu^\pm(z)$ are of the form (near $z=0$)

$$b_\nu^\pm(z)=b_{\nu,1}^\pm(z)+z^{n-2}\log z\,b_{\nu,2}^\pm(z),$$

where $b_{\nu,j}^\pm(z)$ are analytic functions, $b_{\nu,2}^\pm(z)\equiv 0$ if n is odd. Therefore, we have

$$\left|\partial_z^jb_\nu^\pm(z)\right|\le C,\quad 0<z\le 1,\,0\le j\le n-3,$$

$$\left|\partial_z^jb_\nu^\pm(z)\right|\le C_\epsilon z^{-\epsilon},\quad 0<z\le 1,\,j=n-2,$$

$$\left|\partial_z^j b_\nu^\pm(z)\right| \le C_j z^{n-2-j}, \quad 0<z\le 1,\, j\ge n-1,$$

which imply

$$\left|\partial_z^j b_\nu^\pm(z)\right| \le C\langle z\rangle^{(n-3)/2-j}, \quad \forall z>0,\, 0\le j\le n-3, \tag{B.32}$$

$$\left|\partial_z^j b_\nu^\pm(z)\right| \le C_\epsilon z^{-\epsilon}\langle z\rangle^{-(n-1)/2+\epsilon}, \quad \forall z>0,\, j=n-2, \tag{B.33}$$

$$\left|\partial_z^j b_\nu^\pm(z)\right| \le C_j z^{n-2-j}\langle z\rangle^{-(n-1)/2}, \quad \forall z>0,\, j\ge n-1. \tag{B.34}$$

Set

$$A_\pm^{(1)}(\lambda_1,\lambda_2,\sigma_1,\sigma_2) = \lambda_1 e^{\mp i\sigma_1\lambda_1}\partial_{\lambda_1^2}^k\partial_{\lambda_2^2}^m\left(\rho(\lambda_1^2,\lambda_2^2)e^{\pm i\sigma_1\lambda_1}b_\nu^\pm(\sigma_1\lambda_1)\mathcal{J}_\nu(\sigma_2\lambda_2)\right),$$

$$A_\pm^{(2)}(\lambda_1,\lambda_2,\sigma_1,\sigma_2) = \lambda_2 e^{\mp i\sigma_2\lambda_2}\partial_{\lambda_1^2}^m\partial_{\lambda_2^2}^k\left(\rho(\lambda_1^2,\lambda_2^2)\mathcal{J}_\nu(\sigma_1\lambda_1)e^{\pm i\sigma_2\lambda_2}b_\nu^\pm(\sigma_2\lambda_2)\right).$$

By (B.26), (B.30)-(B.34), we have (with $\ell=0,1$)

$$\left|\partial_{\lambda_1}^\ell A_\pm^{(1)}(\lambda_1,\lambda_2,\sigma_1,\sigma_2)\right|$$
$$\le C\langle\sigma_1\rangle^{k+(n-3)/2}\sigma_2^{n-2}\langle\sigma_2\rangle^{m-(n-1)/2}\langle\lambda_1\rangle^{(n-3)/2-k-1}, \quad \forall(\lambda_1,\lambda_2), \tag{B.35}$$

$$\left|\partial_{\lambda_2}^\ell A_\pm^{(2)}(\lambda_1,\lambda_2,\sigma_1,\sigma_2)\right|$$
$$\le C\sigma_1^{n-2}\langle\sigma_1\rangle^{m-(n-1)/2}\langle\sigma_2\rangle^{k+(n-3)/2}\langle\lambda_1\rangle^{(n-3)/2-m-2}, \quad \forall(\lambda_1,\lambda_2). \tag{B.36}$$

Using the inequality

$$\|\widehat{\varphi}(\tau)\|_{L^1} \le C\|\langle\tau\rangle\widehat{\varphi}(\tau)\|_{L^2} \le C\sum_{\ell=0}^1\left\|\partial_\lambda^\ell\varphi(\lambda)\right\|_{L^2} \le C\sum_{\ell=0}^1\sup_\lambda\langle\lambda\rangle\left|\partial_\lambda^\ell\varphi(\lambda)\right|,$$

we obtain from (B.28) and (B.35) (if $k>(n-3)/2$)

$$\left|W_1^{(1)}(\sigma_1,\sigma_2,t,\gamma)\right| \le \sum_\pm Ct^{-k-1/2}\gamma^{-m}(\sigma_1\sigma_2)^{-2\nu}$$
$$\times\int_{-\infty}^\infty\left|\int_0^\infty\int_0^\infty e^{i\tau\lambda_1+i\gamma\lambda_2^2}A_\pm^{(1)}(\lambda_1,\lambda_2,\sigma_1,\sigma_2)d\lambda_1 d\lambda_2^2\right|d\tau$$
$$\le\sum_\pm\sum_{\ell=0}^1 Ct^{-k-1/2}\gamma^{-m}(\sigma_1\sigma_2)^{-2\nu}\sup_{\lambda_1,\lambda_2}\langle\lambda_1\rangle\left|\partial_{\lambda_1}^\ell A_\pm^{(1)}(\lambda_1,\lambda_2,\sigma_1,\sigma_2)\right|$$
$$\le Ct^{-k-1/2}\gamma^{-m}\sigma_1^{-(n-2)/2}\langle\sigma_1\rangle^{k+(n-3)/2}\langle\sigma_2\rangle^{m-(n-1)/2}, \tag{B.37}$$

where we have made a change of variables $\tau\to\tau\pm\sigma_1$. Similarly, by (B.29) and (B.36), we get (if $m>(n-3)/2$)

$$\left|W_1^{(2)}(\sigma_1,\sigma_2,t,\gamma)\right| \le \sum_\pm Ct^{-k-1/2}\gamma^{-m}(\sigma_1\sigma_2)^{-2\nu}$$
$$\times\int_{-\infty}^\infty\left|\int_0^\infty\int_0^\infty e^{i\tau\lambda_2+i\gamma\lambda_1^2}A_\pm^{(2)}(\lambda_1,\lambda_2,\sigma_1,\sigma_2)d\lambda_2 d\lambda_1^2\right|d\tau$$
$$\le\sum_\pm\sum_{\ell=0}^1 Ct^{-k-1/2}\gamma^{-m}(\sigma_1\sigma_2)^{-2\nu}\sup_{\lambda_2}\int_0^\infty\left|\partial_{\lambda_2}^\ell A_\pm^{(2)}(\lambda_1,\lambda_2,\sigma_1,\sigma_2)\right|d\lambda_1^2$$
$$\le Ct^{-k-1/2}\gamma^{-m}\sigma_2^{-(n-2)/2}\langle\sigma_2\rangle^{k+(n-3)/2}\langle\sigma_1\rangle^{m-(n-1)/2}. \tag{B.38}$$

We would like to apply (B.37) and (B.38) with $k=(n-1)/2$, $m=(n-3)/2+\epsilon$, $0<\epsilon\ll 1$. To this end, we need to show that these estimates are valid for all real $(n-3)/2<m\leq(n-2)/2$, $(n-2)/2\leq k<n/2$ if n is even, and for $k=(n-1)/2$ and all real $(n-3)/2<m\leq(n-1)/2$ if n is odd. This can be done by interpolation as follows. Let $\phi\in C_0^\infty(\mathbf{R})$, $\phi(\lambda)=1$ for $|\lambda|\leq 1$, $\phi(\lambda)=0$ for $|\lambda|\geq 2$. Decompose $W_1^{(j)}$ as $X^{(j)}+Y^{(j)}$, $j=1,2$, where $X^{(j)}$ and $Y^{(j)}$ are defined by replacing in the definition of $W_1^{(j)}$ the function ρ by $\phi(\lambda_j)\rho$ and $(1-\phi)(\lambda_j)\rho$, respectively. Clearly, the functions $X^{(j)}$ satisfy (B.37) and (B.38), respectively, for all integers $0\leq k,m<n/2$, while the functions $Y^{(j)}$ satisfy (B.37) and (B.38) for all integers $k>(n-3)/2$, $(n-3)/2<m<n/2$, respectively. When n is odd, this is fulfilled with $k=(n-1)/2$. To show this in the case of even n, we write the function ϕ as

$$\phi(\lambda)=\sum_{p=0}^{\infty}\phi_1(2^p\lambda),$$

with some function $\phi_1\in C_0^\infty(\mathbf{R})$, $\phi_1(\lambda)=0$ in a neighbourhood of $\lambda=0$. Thus,

$$X^{(j)}=\sum_{p=0}^{\infty}X_p^{(j)},$$

where $X_p^{(j)}$ is defined by replacing in the definition of $X^{(j)}$ the function $\phi(\lambda_j)$ by $\phi_1(2^p\lambda_j)$. As above, one can see that the functions $X_p^{(j)}$, $j=1,2$, satisfy (B.37) and (B.38), respectively, with an extra factor in the RHS of the form $2^{p(k-n/2)}$ for all integers $k\geq(n-2)/2$, and hence, by interpolation, for all real $k\geq(n-2)/2$. Therefore, summing up these estimates we conclude that $X^{(j)}$, $j=1,2$, satisfy (B.37) and (B.38), respectively, for all real $(n-2)/2\leq k<n/2$, and in particular for $k=(n-1)/2$. Hence, so do the functions $W_1^{(j)}$. Furthermore, $W_1^{(1)}$ satisfies (B.37) for all integers $0\leq m<n/2$, and hence, by interpolation, for all real $0\leq m\leq(n-1)/2$ if n is odd, and for all real $0\leq m\leq(n-2)/2$ if n is even. In particular, this is valid with $m=(n-3)/2+\epsilon$. To show that the function $W_1^{(2)}$ satisfies (B.38) with $m=(n-3)/2+\epsilon$, we decompose it as $Z+N$, where Z and N are defined by replacing in the definition of $W_1^{(2)}$ the function ρ by $\phi(\lambda_1)\rho$ and $(1-\phi)(\lambda_1)\rho$, respectively. Clearly, the function Z satisfies (B.37) for all integers $0\leq m<n/2$, and hence, by interpolation, for all real $0\leq m\leq(n-1)/2$ if n is odd, and for all real $0\leq m\leq(n-2)/2$ if n is even. To deal with the function N, we write the function $1-\phi$ as

$$(1-\phi)(\lambda)=\sum_{p=0}^{\infty}\phi_2(2^{-p}\lambda),$$

with some function $\phi_2\in C_0^\infty(\mathbf{R})$, $\phi_2(\lambda)=0$ in a neighbourhood of $\lambda=0$. Thus,

$$N=\sum_{p=0}^{\infty}N_p,$$

where N_p is defined by replacing in the definition of N the function $(1-\phi)(\lambda_1)$ by $\phi_2(2^{-p}\lambda_1)$. Now, the functions N_p satisfy (B.38) with an extra factor in the RHS of the form $2^{-p(m-(n-3)/2)}$ for all integers $0\leq m<n/2$, and hence, by interpolation, for all real $0\leq m\leq(n-1)/2$ if n is odd, and for all real $0\leq m\leq(n-2)/2$ if n is even. Therefore, summing up these estimates we conclude that N satisfies (B.38) for all real $(n-3)/2<m\leq(n-1)/2$ if n is odd, and for all real $(n-3)/2<m\leq(n-2)/2$ if n is even. In particular, this is valid with $m=(n-3)/2+\epsilon$.

By (B.37) and (B.38) with $k=(n-1)/2$, $m=(n-3)/2+\epsilon$, we obtain

$$|W_1(\sigma_1,\sigma_2,t,\gamma)|$$

$$\le Ct^{-n/2}\gamma^{-(n-3)/2-\epsilon}\left(\sigma_1^{-n+2}\langle\sigma_1\rangle^{n-2}\langle\sigma_2\rangle^{-1+\epsilon}+\sigma_2^{-n+2}\langle\sigma_2\rangle^{n-2}\langle\sigma_1\rangle^{-1+\epsilon}\right)$$
$$\le Ct^{-n/2}\gamma^{-(n-3)/2-\epsilon}\left(\sigma_1^{-n+2}+\sigma_2^{-n+2}+\langle\sigma_1\rangle^{-1+\epsilon}+\langle\sigma_2\rangle^{-1+\epsilon}\right)$$
$$\le Ct^{-n/2}\gamma^{-(n-3)/2-\epsilon}\left(\sigma_1^{-n+2}+\sigma_2^{-n+2}+\sigma_1^{-1+\epsilon}+\sigma_2^{-1+\epsilon}\right),$$

which is the desired bound. □

Taking $\gamma = h^{\beta'}$ with a suitably chosen constant $\beta' > 0$, we deduce from (B.4), (B.16)-(B.19) and (B.22),

$$\left\|\Psi(t;h)f - F(t)\psi(h^2G_0)f\right\|_{L^\infty}$$
$$\le Ch^{\beta}t^{-n/2}\|f\|_{L^1} + Ch^{\beta}\left\|\Psi(t;h)f - F(t)\psi(h^2G_0)f\right\|_{L^\infty}, \quad \forall f\in L^1, \tag{B.39}$$

with some constant $\beta > 0$. Taking h small enough, we can absorb the second term in the RHS of (B.39), thus obtaining (B.5). □

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